ISRAEL JOURNAL OF MATHEMATICS, Vol. 40, No. 2, 1981

ON THE UNIQUENESS OF SYMMETRIC BASES IN FINITE DIMENSIONAL BANACH SPACES

BY

CARSTEN SCHÜTT[†]

ABSTRACT

Suppose $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ are symmetric bases of the Banach spaces E and F. Let $d(E,F) \leqq C$ and $d(E, l_n^2) \geqq n^r$ for some $r > 0$. Then there is a constant $C_r = C_r(C) > 0$ such that for all $a_i \in R$, $i = 1, \cdots, n$

$$C_r^{-1} \left\| \sum_{i=1}^n a_i e_i \right\| \leqq \left\| \sum_{i=1}^n a_i f_i \right\| \leqq C_r \left\| \sum_{i=1}^n a_i e_i \right\|.$$

We also give a partial uniqueness of unconditional bases under more restrictive conditions.

In the first paragraph we prove that symmetric bases in finite dimensional Banach spaces are unique up to a constant provided we are not "too close" to a Hilbert space. "Too close" is used here in the sense of Banach–Mazur distance. Let $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ be symmetric bases of E and F with $d(E,F) \leqq C$ and $d(E, l_n^2) \geqq n^r$ for some $C, r > 0$. Then there is a constant $C_r = C_r(C) > 0$ such that for all $a_i \in R$, $i = 1, \cdots, n$

$$C_r^{-1} \left\| \sum_{i=1}^n a_i e_i \right\| \leqq \left\| \sum_{i=1}^n a_i f_i \right\| \leqq C_r \left\| \sum_{i=1}^n a_i e_i \right\|.$$

This generalizes a result of Johnson–Maurey–Schechtman–Tzafriri [2]. They proved uniqueness of symmetric bases for spaces with a q-concave basis, $1 \leqq q < 2$. The starting point of their proof was that by 2-concavity the 1-absolutely summing norm of the identity $\pi_1(E)$ is proportional to $\| \Sigma_{i=1}^n e_i \|$ and $\| \Sigma_{i=1}^n f_i \|$. Dropping 2-concavity, we don't know whether $\| \Sigma_{i=1}^n e_i \|$ and $\| \Sigma_{i=1}^n f_i \|$ are proportional. This is the main difficulty we have to deal with although it does not appear explicitly in the proof.

[†] During the preparation of this paper the author was supported by the Deutsche Forschungsgemeinschaft (DFG).

Received November 26, 1980 and in revised form March 29, 1981

Although our proof differs completely from the proof of Johnson–Maurey–Schechtman–Tzafriri there is a common point. Consider the matrix of the map $A \in L(E, F)$ with $\|A\| \|A^{-1}\| = d(E, F)$ with respect to the bases $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$. Then, roughly speaking, we prove that the matrix has a "big" diagonal.

This is also the reason why the proof does not work if we come "too close" to a Hilbert space. In a Hilbert space all orthogonal bases are symmetric. By now, there are a lot of facts suggesting that we have uniqueness of symmetric bases in general (in finite dimensional spaces). On the other hand, we would not be too surprised if there is a counterexample.

In the second paragraph we consider unconditional bases. It is known [2] that there is no uniqueness of unconditional bases. But we discover a partial uniqueness, i.e., a certain percentage of the basis has to be unique.

Let $\{e_i\}_{i=1}^n$ be a symmetric and $\{f_i\}_{i=1}^n$ an unconditional basis of E and F. Suppose $d(E, F) \leqq C$, $\text{cotype}_2(E) \leqq C$ and one of the bases dominates the unit vector basis in l_n^p, $1 \leqq p < 2$. Then there are $C^* = C^*(C, p) > 0$, $\varepsilon = \varepsilon(C, p) > 0$ and a subset M of $\{1, \cdots, n\}$ with $|M| \geqq \varepsilon n$ and

$$C^{*-1} \left\| \sum_{i \in M} a_i e_i \right\| \leqq \left\| \sum_{i \in M} a_i f_i \right\| \leqq C^* \left\| \sum_{i \in M} a_i e_i \right\|$$

for all $a_i \in R$, $i = 1, \cdots, n$.

This is used in order to estimate the symmetric basis constant $\text{sbc}(E)$ of certain spaces. In [2] it was pointed out that if E has an unconditional basis $\{e_i\}_{i=1}^n$ such that for all $a_i \in R$, $i = 1, \cdots, n$

$$\left(\sum_{i=1}^n |a_i|^2 \right)^{1/2} \leqq \left\| \sum_{i=1}^n a_i e_i \right\|$$

then one has

$$\frac{2}{n} \left\| \sum_{i=1}^n e_i \right\| \left\| \sum_{i=1}^n e_i^* \right\| \leqq \text{s}(E)^2 \leqq \text{sbc}(E)^2$$

where $\text{s}(E)$ denotes the symmetry constant introduced by Garling and Gordon [1]. But for spaces $l_n^p \otimes_r l_n^r$

$$\|(a_{ij})_{ij=1}^n\| = \left(\sum_{i=1}^n \left(\sum_{j=1}^n |a_{ij}|^p \right)^{r/p} \right)^{1/r}$$

we have

$$\text{s}(l_n^p \otimes_r l_n^r) = 1$$

so that this cannot be used to estimate $\mathrm{sbc}(l_n^p \otimes_r l_n^r)$. We get estimations for $\mathrm{sbc}(l_n^p \otimes_r l_n^r)$.

Thus we give also examples of sequences of spaces E_n, $n \in N$, such that $s(E_n) = \mathrm{ubc}(E_n) = 1$ but $\mathrm{sbc}(E_n)$ tends to infinity. The first example of this kind was given in [3].

0. Notation

The symmetric basis constant $\mathrm{sbc}(\{e_i\}_{i=1}^n)$ of a basis $\{e_i\}_{i=1}^n$ of a Banach space E is the infimum of all numbers $C > 0$ such that

$$\left\| \sum_{i=1}^n a_i e_i \right\| \leqq C \left\| \sum_{i=1}^n \varepsilon_i a_i e_{\pi(i)} \right\|$$

for all signs $\varepsilon_i = \pm 1$, all $a_i \in R$, $i = 1, \cdots, n$, and all permutations π of $\{1, \cdots, n\}$. The unconditional basis constant $\mathrm{ubc}(\{e_i\}_{i=1}^n)$ is the infimum of all numbers $C > 0$ such that

$$\left\| \sum_{i=1}^n a_i e_i \right\| \leqq C \left\| \sum_{i=1}^n \varepsilon_i a_i e_i \right\|$$

for all signs $\varepsilon_i = \pm 1$, and all $a_i \in R$, $i = 1, \cdots, n$. We put

$$\mathrm{sbc}(E) = \inf\{\mathrm{sbc}(\{e_i\}_{i=1}^n) \mid \{e_i\}_{i=1}^n \text{ is a basis}\};$$

$\mathrm{ubc}(E)$ is defined analogously.

We say that E has a symmetric (unconditional) basis $\{e_i\}_{i=1}^n$ if it is normalized and $\mathrm{sbc}(\{e_i\}_{i=1}^n) = 1$ ($\mathrm{ubc}(\{e_i\}_{i=1}^n) = 1$).

The Banach–Mazur distance of two Banach spaces E and F is given by

$$d(E, F) = \inf\{\|I\| \|I^{-1}\| \mid I \in L(E, F)\}.$$

The 1-absolutely summing norm of an operator $A \in L(E, F)$ is given by the infimum of all constants $C > 0$ such that for all sequences $\{x_i\}_{i=1}^k$ we have

$$\sum_{i=1}^k \|A(x_i)\| \leqq C \sup_{\|x^*\|=1} \sum_{i=1}^k |\langle x_i, x^* \rangle|.$$

We say that a space has cotype p, $2 \leqq p < \infty$, with constant $C > 0$ if for all sequences $\{x_i\}_{i=1}^k$ we have

$$\left(\sum_{i=1}^k \|x_i\|^p \right)^{1/p} \leqq C 2^{-k} \sum_{\varepsilon} \left\| \sum_{i=1}^k \varepsilon_i x_i \right\|$$

where $\varepsilon_i = \pm 1$, $i = 1, \cdots, k$.

Of course, we use the Khintchine inequality [4] frequently,

$$\frac{1}{\sqrt{2}}\left(\sum_{i=1}^{n}|a_i|^2\right)^{1/2} \leqq 2^{-n}\sum_{\varepsilon}\left|\sum_{i=1}^{n}\varepsilon_i a_i\right| \leqq \left(\sum_{i=1}^{n}|a_i|^2\right)^{1/2},$$

where $\varepsilon_i = \pm 1$, $i = 1,\cdots,n$.

By $|M|$ or card M we mean the cardinality of a set M. The entries of the inverse matrix A^{-1} are denoted by b_{ij}, $i,j = 1,\cdots,n$. $[r]$ is the greatest natural number smaller than r, $r \geqq 1$. If $r<1$ we put $[r]=1$.

By $\{e_i^*\}_{i=1}^n$ we denote the dual basis of $\{e_i\}_{i=1}^n$.

1. On the uniqueness of symmetric bases

We prove the following theorem.

THEOREM 1.1. *Let E and F be Banach spaces with symmetric bases $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$. Suppose that $d(E,F)\leqq C$ and $d(E,l_n^2)\geqq n^r$ for some $C>0$ and $r>0$. Then there is a constant $C_r = C_r(C)>0$ such that for all $a_i \in R$, $i=1,\cdots,n$ we have*

$$C_r^{-1}\left\|\sum_{i=1}^{n}a_i e_i\right\| \leqq \left\|\sum_{i=1}^{n}a_i f_i\right\| \leqq C_r\left\|\sum_{i=1}^{n}a_i e_i\right\|.$$

For the proof of this theorem we need several lemmas. We start by "slicing" matrices. Let $A=\{a_{ij}\}_{i,j=1}^n$ be a matrix and $A^{-1}=\{b_{ij}\}_{i,j=1}^n$ its inverse. Suppose that $|a_{ij}|\leqq C_1$ and $|b_{ji}|\leqq C_2$ for all $i,j=1,\cdots,n$. Then we define

$$J_i^+ = \{j \mid \operatorname{sign} a_{ij} = \operatorname{sign} b_{ji}\},$$

$$A_i^{k,l} = \{j \in J_i^+ \mid C_1 2^{-k} < |a_{ij}| \leqq C_1 2^{-k+1}, C_2 2^{-l} < |b_{ji}| \leqq C_2 2^{-l+1}\}$$

$$\text{for } k,l = 1,2,3,\cdots.$$

LEMMA 1.2. *Let A be a real-valued $n\times n$-matrix and A^{-1} its inverse. Then there are $k,l \in N$ and a subset M of $\{1,\cdots,n\}$ such that*

(i) $|M| \geqq \frac{1}{4}n(kl)^{-2}$,

(ii) $\sum_{j\in A_i^{k,l}} a_{ij}b_{ji} \geqq \frac{1}{4}(kl)^{-2}$ *for all* $i \in M$.

PROOF. We observe first that for all $i=1,\cdots,n$ there are k and l such that

$$\frac{1}{4}(kl)^{-2} \leqq \sum_{j\in A_i^{k,l}} a_{ij}b_{ji}. \tag{1.1}$$

Suppose this is not true. Because of $\mathrm{id} = AA^{-1}$, $|a_{ij}|\leqq C_1$ and $|b_{ji}|\leqq C_2$ for all $i,j=1,\cdots,n$ we get

$$1 \leqq \sum_{j \in J_i^+} a_{ij} b_{ji} \leqq \sum_{k,l=1}^{\infty} \sum_{j \in A_i^{k,l}} a_{ij} b_{ji}.$$

Since we assume the opposite of (1.1) we get

$$1 \leqq \sum_{k,l=1}^{\infty} \frac{1}{4}(kl)^{-2} = \frac{1}{4}\left(\frac{\pi^2}{6}\right)^2 < 1.$$

Thus (1.1) is true. So we choose from every row a set $A_i^{k(i),l(i)}$ satisfying (1.1). We group the rows with the same k, l,

$$M(k,l) = \{i \mid k = k(i) \text{ and } l = l(i)\}.$$

There is at least one set $M(k,l)$ with $|M(k,l)| \geqq \frac{1}{4} n (kl)^{-2}$. If not, we get a contradiction:

$$n = \operatorname{card}\left\{ \bigcup_{k,l=1}^{\infty} M(k,l) \right\} \leqq \sum_{k,l=1}^{\infty} |M(k,l)| \leqq \frac{1}{4} n \sum_{k,l=1}^{\infty} (kl)^{-2} < n.$$

Of course, we choose as our set M the set $M(k,l)$ satisfying $|M(k,l)| \geqq \frac{1}{4} n (kl)^{-2}$. □

LEMMA 1.3. *Suppose A is a $m \times n$-matrix, $m \leqq n$, where in every row there are k entries equal to one, the others are zero. By A_i we denote the subset of $\{1, \cdots, n\}$ of indices of those entries of the i'th row that are one. Then we find for every j, $1 \leqq j \leqq n$, a subset J of $\{1, \cdots, n\}$ with $|J| \leqq j$ and for*

(i) $j = 1, \cdots, [n/k]$

$$\operatorname{card}\{i \mid J \cap A_i \neq \emptyset\} \geqq \frac{1}{4} \frac{m}{n} jk;$$

(ii) $j = [n/k], \cdots, n$, *at least* $\frac{1}{128} m^2/n$ *rows, i.e., indices i, satisfy*

$$\operatorname{card}\{A_i \cap J\} \geqq \frac{1}{260} \frac{m}{n} \frac{jk}{n}.$$

PROOF. We define the set J in (i) inductively. There is a sequence $(j_t, M_t)_{t=1}^{l}$, $l \leqq j$, such that $j_t \in \{1, \cdots, n\}$, $M_t \subset \{1, \cdots, m\}$ with

$$\tag{1.2} \begin{aligned} &m_1 = |M_1| \geqq mk/n, \\ &m_r = \operatorname{card}\left\{ M_r \setminus \bigcup_{t=1}^{r-1} M_t \right\} \geqq \frac{1}{2} \frac{m}{n} k \quad \text{for all } r = 2, \cdots, l; \end{aligned}$$

$$\tag{1.3} a_{s,j_t} = \begin{cases} 1 & \text{for } s \in M_t \\ 0 & \text{for } s \notin M_t \end{cases} \quad \text{for } t = 1, \cdots, l;$$

$$\sum_{r=1}^{l-1} m_r \leqq \frac{1}{4}\frac{m}{n} jk < \sum_{r=1}^{l} m_r. \tag{1.4}$$

If we have constructed this we have proved (i). Indeed, we choose as $J = \{j_1, \cdots, j_l\}$ and we have

$$\operatorname{card}\{i \mid J \cap A_i \neq \varnothing\} = \operatorname{card}\left\{ \bigcup_{t=1}^{l} M_t \right\}$$

$$\geqq \sum_{r=1}^{l} \operatorname{card}\left\{ M_r \setminus \bigcup_{t=1}^{r-1} M_t \right\} \geqq \sum_{r=1}^{l} m_r \geqq \frac{1}{4}\frac{m}{n} jk.$$

Now we prove the construction. Since A contains km ones this makes mk/n ones in the average per column. We choose a column j_1 and a set M_1 with $|M_1| = m_1 \geqq mk/n$. If $m_1 > mjk/4n$ we are done. If not, assume we have already chosen j_{r-1}, M_{r-1} and we choose now j_r and M_r. By (1.4) we know that

$$\sum_{t=1}^{r-1} m_t \leqq \frac{1}{4} m.$$

Therefore we have

$$\operatorname{card}\left\{ \{1, \cdots, m\} \setminus \left\{ \bigcup_{t=1}^{r-1} M_t \right\} \right\} \geqq \frac{1}{2} m.$$

Thus we have in the average in the submatrix defined by the index set

$$\left\{ \{1, \cdots, m\} \setminus \left\{ \bigcup_{t=1}^{r-1} M_t \right\} \right\} \times \{1, \cdots, n\}$$

at least $mk/2n$ ones per column. We choose a column j_r in which more than $\frac{1}{2}mk/n$ ones can be found. At last, we have to check that $l \leqq j$:

$$\frac{1}{2}\frac{m}{n} k(l-1) \leqq \sum_{t=1}^{l-1} m_t \leqq \frac{1}{4}\frac{m}{n} jk.$$

Thus $2(l-1) \leqq j$.

The proof of (ii) is essentially a repeated application of (i). In order to prove (ii) we assume the converse and construct a contradiction.

Assume now that (ii) is not true. We construct mutually disjoint subsets J_l of $\{1, \cdots, n\}$ and subsets N_l of $\{1, \cdots, m\}$, $l = 1, \cdots, [jk/n]$, defined by

$$N_l = \left\{ i \mid \operatorname{card}\left\{ \left\{ \bigcup_{t=1}^{l} J_t \right\} \cap A_i \right\} > \frac{1}{260}\frac{m}{n}\frac{jk}{n} \right\}$$

such that

$$(1.5)\qquad |N_l| \leqq \frac{1}{128}\frac{m^2}{n} \qquad \text{for all } l = 1, \cdots, \left[\frac{jk}{n}\right],$$

$$(1.6)\qquad \operatorname{card}\{i \mid J_l \cap A_i \neq \emptyset\} \geqq \frac{1}{64}\frac{m^2}{n},$$

$$(1.7)\qquad \operatorname{card}\{i \mid J_l \cap A_i \neq \emptyset \text{ and } i \notin N_{[jk/n]}\} \geqq \frac{1}{128}\frac{m^2}{n},$$

$$(1.8)\qquad |J_l| \leqq m/k \qquad \text{for all } l = 1, \cdots, [jk/n],$$

$$(1.9)\qquad \operatorname{card}\left\{\bigcup_{l=1}^{[jk/n]} J_l\right\} \leqq j.$$

Two of these conditions are immediately verified: (1.9) follows from (1.8), (1.7) follows from (1.5) and (1.6).

We get J_1 by applying (i) with $k' = k$ and $j' = [m/k]$. Thus we have $|J_1| \leqq m/k$ and

$$\operatorname{card}\{i \mid J_1 \cap A_i \neq \emptyset\} \geqq \frac{1}{4}\frac{m}{n}k\left[\frac{m}{k}\right] \geqq \frac{1}{8}\frac{m^2}{n}$$

thus fulfilling (1.6) and (1.8). Clearly (1.5) is also fulfilled. If not, (ii) would be true with $J = J_1$. But we assume the opposite of (ii). Assume now we have already chosen $r-1$ sets J_l and choose now the set J_r. We consider the submatrix described by the index set

$$(1.10)\qquad \{\{1, \cdots, m\}\backslash N_{r-1}\} \times \left\{\{1, \cdots, n\}\backslash \left\{\bigcup_{l=1}^{r-1} J_l\right\}\right\}.$$

By the definition of N_{r-1} we get for all $i \notin N_{r-1}$

$$\operatorname{card}\left\{\left\{\{1, \cdots, n\}\backslash \bigcup_{l=1}^{r-1} J_l\right\} \cap A_i\right\} \geqq k/2$$

since $|A_i| = k$ for all $i = 1, \cdots, m$. Therefore, every row of this submatrix contains at least $k/2$ ones. We note also that we have because of $m \leqq n$ and (1.5)

$$\operatorname{card}\{\{1, \cdots, m\}\backslash N_{r-1}\} \geqq m - \frac{1}{64}\frac{m^2}{n} > \frac{1}{2}m.$$

We apply now (i) to this submatrix (1.10) with $k' = [k/2]$ and $j' = [m/k]$. We get a set J_r with $|J_r| \leqq \frac{1}{2}m/k$ and

$$\operatorname{card}\{i \mid J_r \cap A_i \neq \emptyset\} \geqq \frac{1}{8}\frac{m}{n}\left[\frac{k}{2}\right]\left[\frac{m}{k}\right] \geqq \frac{1}{64}\frac{m^2}{n}.$$

Again, $|N_r| \leqq \frac{1}{128} m^2/n$ since we assume the opposite of (ii). Else, (ii) would be satisfied for $J = \bigcup_{l=1}^{r} J_l$.

It is obvious that we can continue this process at least until $r = [jk/n]$. Considering now the sets J_l as constructed above we derive a contradiction.

We consider the submatrix described by the index set

$$\left\{ \bigcup_{l=1}^{[jk/n]} J_l \right\} \times \{\{1, \cdots, m\} \backslash N_{[jk/n]}\}. \tag{1.11}$$

We know by the definition of $N_{[jk/n]}$ that every row of this submatrix contains less than $\frac{1}{260}(m/n)(jk/n)$ ones. On the other hand, we claim that there is a row in (1.11) having more than $\frac{1}{256}(m/n)(jk/n)$ ones, thus giving a contradiction.

Indeed, by (1.7) every group of columns contains at least $\frac{1}{128} m^2/n$ ones. This makes

$$\left[\frac{jk}{n}\right] \frac{1}{128} \frac{m^2}{n} \geqq \frac{1}{256} \frac{m^2}{n} \frac{jk}{n}$$

ones in all. Since we have at most m rows we have more than $\frac{1}{256}(m/n)(jk/n)$ ones in the average per row. □

LEMMA 1.4. *Suppose E and F have symmetric bases* $\{e_i\}_{i=1}^n$ *and* $\{f_i\}_{i=1}^n$ *resp. Let* $A \in L(E, F)$. *Then*

$$\|A\| \geqq \frac{1}{\sqrt{2}} \max_{l=1,\cdots,n} \max_{\pi} \left\| \sum_{i=1}^{l} e_i \right\|^{-1} \left\| \sum_{j=1}^{n} \left(\sum_{i=1}^{l} |\langle A(e_{\pi(i)}), f_j^* \rangle|^2 \right)^{1/2} f_j \right\|$$

where π *varies over all permutations of* $\{1, \cdots, n\}$.

PROOF. By using the Khintchine inequality and the triangle inequality we get

$$\|A\| \geqq \max_{l=1,\cdots,n} \max_{\pi} \left\| \sum_{i=1}^{l} e_i \right\|^{-1} \left\| \sum_{j=1}^{n} \left| \sum_{i=1}^{l} \pm \langle A(e_{\pi(i)}), f_j^* \rangle \right| f_j \right\|$$

$$\geqq \frac{1}{\sqrt{2}} \max_{l=1,\cdots,n} \max_{\pi} \left\| \sum_{i=1}^{l} e_i \right\|^{-1} \left\| \sum_{j=1}^{n} \left(\sum_{i=1}^{l} |\langle A(e_{\pi(i)}), f_j^* \rangle|^2 \right)^{1/2} f_j \right\|. \quad □$$

LEMMA 1.5. *Let* $\{e_i\}_{i=1}^n$ *and* $\{f_i\}_{i=1}^n$ *be symmetric bases of E and F. Let* $A \in L(E, F)$ *and suppose that in the matrix given by A there are m rows*

$$(\langle A(e_j), f_i^* \rangle)_{j=1}^n$$

having at least r entries, equal to one. Then for

(i) $t = 1, \cdots, [n/r]$

$$\|A\| \geqq C \frac{m}{n} \left\| \sum_{i=1}^{t} e_i \right\|^{-1} \left\| \sum_{i=1}^{tr} f_i \right\|,$$

(ii) $t=[n/r],\cdots,n$

$$\|A\|\geqq C\left(\frac{m}{n}\right)^{5/2}\sqrt{\frac{tr}{n}}\left\|\sum_{i=1}^{t}e_i\right\|^{-1}\left\|\sum_{i=1}^{n}f_i\right\|,$$

where $C>0$ is an absolute constant.

PROOF. We use Lemmas 1.3 and 1.4. We choose a subset J, $|J|\leqq t$, of $\{1,\cdots,n\}$ such that Lemma 1.3(i) is satisfied. Choosing a proper permutation π so that $J\subset\{\pi(\alpha)\,|\,\alpha=1,\cdots,t\}$ we apply Lemma 1.4. We get for some $C>0$

$$\|A\|\geqq C\left\|\sum_{i=1}^{t}e_i\right\|^{-1}\left\|\sum_{i=1}^{[mtr/4n]}f_i\right\|$$

$$\geqq C'\frac{m}{n}\left\|\sum_{i=1}^{t}e_i\right\|^{-1}\left\|\sum_{i=1}^{tr}f_i\right\|.$$

Since we have this for all $t=1,\cdots,[n/r]$ we have proved (i). (ii) follows analogously. □

LEMMA 1.6. *Let $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ be symmetric bases of E and F. Let $A\in L(E,F)$ with $\|A\|=1$, $\|A^{-1}\|\leqq C$. Consider $k,l\in N$ (resp. $k',l'\in N$) given by Lemma 1.2 for the matrix*

$$(\langle A(e_j),f_i^*\rangle)_{i,j=1}^n\qquad (resp.\ (\langle A^{-1}(f_j),e_i^*\rangle)_{i,j=1}^n).$$

Then there is an absolute constant $C^*=C^*(C)>0$ such that for

(i) $t=1,\cdots,[n2^{-k-l}]$

$$C^{*-1}(kl)^{-8}\left\|\sum_{i=1}^{t2^{k+l}}f_i\right\|\leqq\left\|\sum_{i=1}^{2^{k+l}}f_i\right\|\left\|\sum_{i=1}^{t}e_i\right\|\leqq C^*(kl)^8\left\|\sum_{i=1}^{t2^{k+l}}f_i\right\|,$$

(ii) $t=[n2^{-k-l}],\cdots,n$

$$C^{*-1}(kl)^{-12}\left\|\sum_{i=1}^{n}f_i\right\|\leqq\sqrt{nt^{-1}2^{-k-l}}\left\|\sum_{i=1}^{2^{k+l}}f_i\right\|\left\|\sum_{i=1}^{t}e_i\right\|$$

$$\leqq C^*(kl)^{12}\left\|\sum_{i=1}^{n}f_i\right\|,$$

(iii) $t=1,\cdots,[n2^{-k'-l'}]$

$$C^{*-1}(k'l')^{-8}\left\|\sum_{i=1}^{t2^{k'+l'}}e_i\right\|\leqq\left\|\sum_{i=1}^{2^{k'+l'}}e_i\right\|\left\|\sum_{i=1}^{t}f_i\right\|$$

$$\leqq C^*(k'l')^8\left\|\sum_{i=1}^{t2^{k'+l'}}e_i\right\|,$$

(iv) $t=[n2^{-k'-l'}],\cdots,n$

$$C^{*-1}(k'l')^{-12}\left\|\sum_{i=1}^{n}e_i\right\|\leqq\sqrt{nt^{-1}2^{-k'-l'}}\left\|\sum_{i=1}^{2^{k'+l'}}e_i\right\|\left\|\sum_{i=1}^{t}f_i\right\|$$

$$\leqq C^*(k'l')^{12}\left\|\sum_{i=1}^{n}e_i\right\|,$$

(v)

$$C^{*-1}(klk'l')^{-12}2^{(k+l+k'+l')/2}\leqq\left\|\sum_{i=1}^{2^{k'+l'}}e_i\right\|\left\|\sum_{i=1}^{2^{k+l}}f_i\right\|$$

$$\leqq C^*(klk'l')^{12}2^{(k+l+k'+l')/2}.$$

PROOF. Since $\|A\|=1$ and $\|A^{-1}\|\leqq C$ we have

$$|\langle A(e_j),f_i^*\rangle|\leqq 1\quad\text{and}\quad|\langle A^{-1}(f_i),e_j^*\rangle|\leqq C$$

for all $i,j=1,\cdots,n$. Thus we get by Lemma 1.2 that there are $k,l\in N$ and more than $\frac{1}{4}n(kl)^{-2}$ rows, i.e., indices i, with

$$\sum_{j\in A_i^{k,l}}\langle A(e_j),f_i^*\rangle\langle A^{-1}(f_i),e_j^*\rangle\geqq\frac{1}{4}(kl)^{-2}$$

and

$$2^{-k}<|\langle A(e_j),f_i^*\rangle|\leqq 2^{-k+1},\quad C2^{-l}<|\langle A^{-1}(f_i),e_j^*\rangle|\leqq C2^{-l+1}$$

for all $j\in A_i^{k,l}$. In particular we get

$$\frac{1}{4}C^{-1}(kl)^{-2}2^{k+l}\leqq|A_i^{k,l}|. \tag{1.12}$$

Now we apply Lemma 1.5 to A with $m=[\frac{1}{4}n(kl)^{-2}]$ and $r=[\frac{1}{4}C^{-1}(kl)^{-2}2^{k+l}]$. After using the triangle inequality we get for a constant $C'=C'(C)>0$

$$1=\|A\|\geqq C'2^{-k}(kl)^{-4}\left\|\sum_{i=1}^{t}e_i\right\|^{-1}\left\|\sum_{i=1}^{t2^{k+l}}f_i\right\| \quad \text{for } t=1,\cdots,[n2^{-k-l}], \tag{1.13}$$

$$1=\|A\|\geqq C'2^{-k}(kl)^{-8}\sqrt{t2^{k+l}n^{-1}}\left\|\sum_{i=1}^{t}e_i\right\|^{-1}\left\|\sum_{i=1}^{n}f_i\right\| \quad \text{for } t=[n2^{-k-l}],\cdots,n, \tag{1.14}$$

$$C\geqq\|A^{-1}\|\geqq C'2^{-l}(kl)^{-4}\left\|\sum_{i=1}^{t}e_i^*\right\|^{-1}\left\|\sum_{i=1}^{t2^{k+l}}f_i^*\right\| \quad \text{for } t=1,\cdots,[n2^{-k-l}] \tag{1.15}$$

$$C \geqq \| A^{-1} \| \geqq C' 2^{-l} (kl)^{-8} \sqrt{t 2^{k+l} n^{-1}} \left\| \sum_{i=1}^{t} e_i^* \right\|^{-1} \left\| \sum_{i=1}^{n} f_i^* \right\| \tag{1.16}$$

$$\text{for } t = [n 2^{-k-l}], \cdots, n.$$

For $t = 1$ we get from (1.13) and (1.15)

$$C'^{-1} 2^k (kl)^4 \geqq \left\| \sum_{i=1}^{2^{k+l}} f_i \right\| \quad \text{and} \quad CC'^{-1} 2^l (kl)^4 \geqq \left\| \sum_{i=1}^{2^{k+l}} f_i^* \right\|.$$

By duality we get

$$C'^{-1} 2^{-l} (kl)^4 \geqq \left\| \sum_{i=1}^{2^{k+l}} f_i^* \right\|^{-1} \quad \text{and} \quad CC'^{-1} 2^{-k} (kl)^4 \geqq \left\| \sum_{i=1}^{2^{k+l}} f_i \right\|^{-1}.$$

Putting this into the formulas (1.13)–(1.16) we get (i) and (ii). Indeed, the left hand side inequalities follow from (1.13) and (1.14) and the right hand side inequalities are obtained from (1.15) and (1.16) by dualization. Clearly, (iii) and (iv) are achieved the same way.

We prove (v). By (ii) we get for $t = n$

$$C^{*-1} (kl)^{-12} \sqrt{2^{k+l}} \leqq \left\| \sum_{i=1}^{n} e_i \right\| \left\| \sum_{i=1}^{n} f_i \right\|^{-1} \left\| \sum_{i=1}^{2^{k+l}} f_i \right\|$$

$$\leqq C^* (kl)^{12} \sqrt{2^{k+l}}.$$

And by (iv) we get for $t = n$

$$C^{*-1} (k'l')^{-12} \sqrt{2^{k'+l'}} \leqq \left\| \sum_{i=1}^{n} f_i \right\| \left\| \sum_{i=1}^{n} e_i \right\|^{-1} \left\| \sum_{i=1}^{2^{k'+l'}} e_i \right\|$$

$$\leqq C^* (kl)^{12} \sqrt{2^{k'+l'}}.$$

By multiplying these inequalities we get (v). □

LEMMA 1.7. *Let $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ be symmetric bases of E and F. Let $A \in L(E, F)$ with $\|A\| = 1$, $\|A^{-1}\| \leqq C$. Consider $k, l \in N$ (resp. $k', l' \in N$) given by Lemma 1.2 for the matrix*

$$(\langle A(e_j), f_i^* \rangle)_{i,j=1}^n \qquad (\textit{resp. } (\langle A^{-1}(f_j), e_i^* \rangle)_{i,j=1}^n).$$

Then there is an absolute constant $C^* = C^*(C) > 0$ such that

(i) $j = 1, \cdots, [n 2^{-k-l-k'-l'}]$

$$C^{*-1} (klk'l')^{-20} \left\| \sum_{i=1}^{j} e_i \right\| \leqq 2^{-(k+l+k'+l')/2} \left\| \sum_{i=1}^{j 2^{k+l+k'+l'}} e_i \right\|$$

$$\leqq C^* (klk'l')^{20} \left\| \sum_{i=1}^{j} e_i \right\|,$$

(ii) $j=[n2^{-k-l-k'-l'}],\cdots,n$

$$C^{*-1}(klk'l')^{-36}\left\|\sum_{i=1}^{j}e_i\right\|\leqq\sqrt{\frac{j}{n}}\left\|\sum_{i=1}^{n}e_i\right\|\leqq C^*(klk'l')^{36}\left\|\sum_{i=1}^{j}e_i\right\|,$$

(iii) $j=1,\cdots,n$

$$C^{*-t}(klk'l')^{-t\beta}\sqrt{j}\leqq\left\|\sum_{i=1}^{j}e_i\right\|\leqq C^{*t}(klk'l')^{t\beta}\sqrt{j},$$

where t satisfies $2^{t(k+l+k'+l')}\geqq n$ *and* $\beta>0$ *is an absolute constant. The same inequalities hold for the basis* $\{f_i\}_{i=1}^n$.

Proof. For simplicity of notation we introduce $r=2^{k+l}$, $s=2^{k'+l'}$. By Lemma 1.6(i) and (iii) we get

$$C^{*-1}(kl)^{-8}\left\|\sum_{i=1}^{j}e_i\right\|\leqq\left\|\sum_{i=1}^{r}f_i\right\|^{-1}\left\|\sum_{i=1}^{jr}f_i\right\|\quad\text{for } j=1,\cdots,[n/r],$$

$$C^{*-1}(k'l')^{-8}\left\|\sum_{i=1}^{jr}f_i\right\|\leqq\left\|\sum_{i=1}^{s}e_i\right\|^{-1}\left\|\sum_{i=1}^{jrs}e_i\right\|\quad\text{for } j=1,\cdots,[n/rs].$$

Combining these two inequalities we get for all $j=1,\cdots,[n/rs]$

$$C^{*-2}(klk'l')^{-8}\left\|\sum_{i=1}^{j}e_i\right\|\leqq\left\|\sum_{i=1}^{r}f_i\right\|^{-1}\left\|\sum_{i=1}^{s}e_i\right\|^{-1}\left\|\sum_{i=1}^{jrs}e_i\right\|.$$

By Lemma 1.6(v) we get now the left hand side inequality of (i). The right hand side inequality is achieved in the same way.

We prove now (ii). We prove that for $j=[n/rs],\cdots,n$ we have for some $C^*>0$,

$$C^{*-2}(klk'l')^{-24}\left\|\sum_{i=1}^{j}e_i\right\|$$

$$\leqq\sqrt{jrsn^{-1}}\left\|\sum_{i=1}^{r}f_i\right\|^{-1}\left\|\sum_{i=1}^{s}e_i\right\|^{-1}\left\|\sum_{i=1}^{n}e_i\right\|\tag{1.17}$$

$$\leqq C^{*2}(klk'l')^{24}\left\|\sum_{i=1}^{j}e_i\right\|.$$

From these inequalities and Lemma 1.6(v) the inequalities (ii) follow. We have to consider two cases, $j=[n/rs],\cdots,[n/r]$ and $j=[n/r],\cdots,n$. If $j=[n/rs],\cdots,[n/r]$ we get by Lemma 1.6(i) and (iv)

$$C^{*-1}(kl)^{-8}\left\|\sum_{i=1}^{j} e_i\right\| \leqq \left\|\sum_{i=1}^{r} f_i\right\|^{-1}\left\|\sum_{i=1}^{jr} f_i\right\|,$$

$$C^{*-1}(kl)^{-12}\left\|\sum_{i=1}^{jr} f_i\right\| \leqq \sqrt{jrsn^{-1}}\left\|\sum_{i=1}^{s} e_i\right\|^{-1}\left\|\sum_{i=1}^{n} e_i\right\|.$$

Combining these two inequalities we get the left hand side inequality of (1.17). The right hand side inequality is established in the same way.

If $j = [n/r], \cdots, n$ we have by Lemma 1.6(ii) and (iv) for $t = n$

$$C^{*-1}(kl)^{-12}\left\|\sum_{i=1}^{j} e_i\right\| \leqq \sqrt{jrn^{-1}}\left\|\sum_{i=1}^{r} f_i\right\|^{-1}\left\|\sum_{i=1}^{n} f_i\right\|,$$

$$C^{*-1}(k'l')^{-12}\left\|\sum_{i=1}^{n} f_i\right\| \leqq \sqrt{s}\left\|\sum_{i=1}^{s} e_i\right\|^{-1}\left\|\sum_{i=1}^{n} e_i\right\|.$$

Combining these two inequalities we get the left hand side inequality of (1.17). Again, the right hand side inequality is obtained in the same way.

We prove now (iii). We prove that for t with $n \leqq 2^{t(k+l+k'+l')}$ and all $j = 1, \cdots, n$ we have

$$\begin{aligned} C^{*-t}(klk'l')^{-56t}\left\|\sum_{i=1}^{j} e_i\right\| &\leqq \sqrt{\frac{j}{n}}\left\|\sum_{i=1}^{n} e_i\right\| \\ &\leqq C^{*t}(klk'l')^{56t}\left\|\sum_{i=1}^{j} e_i\right\|. \end{aligned} \tag{1.18}$$

From (1.18) we get immediately (iii). Indeed, (1.18) gives for $j = 1$

$$C^{*-t}(klk'l')^{-56t}\sqrt{n} \leqq \left\|\sum_{i=1}^{n} e_i\right\| \leqq C^{*t}(klk'l')^{56t}\sqrt{n}.$$

Putting these inequalities into (1.18) gives (iii). We verify (1.18). Clearly, by (ii) we get (1.18) for $j = [n(rs)^{-1}], \cdots, n$. We get by (i) and (ii) for all $j = [n(rs)^{-2}], \cdots, [n(rs)^{-1}]$

$$C^{*-1}(klk'l')^{-20}\left\|\sum_{i=1}^{j} e_i\right\| \leqq \sqrt{\frac{1}{rs}}\left\|\sum_{i=1}^{jrs} e_i\right\|,$$

$$C^{*-1}(klk'l')^{-36}\left\|\sum_{i=1}^{jrs} e_i\right\| \leqq \sqrt{jrsn^{-1}}\left\|\sum_{i=1}^{n} e_i\right\|.$$

Combining these two inequalities we get for $j = [n(rs)^{-2}], \cdots, [n(rs)^{-1}]$

$$C^{*-2}(klk'l')^{-56}\left\|\sum_{i=1}^{j} e_i\right\| \leqq \sqrt{\frac{j}{n}}\left\|\sum_{i=1}^{n} e_i\right\|.$$

This process is repeated t times such that $n \leqq (rs)^t$. Thus we have the left hand side inequality of (1.18). The right hand side inequality is established in the same way. □

Lemma 1.8. *Let E and F be Banach space with symmetric bases $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ such that for some $C>0$ and all $j=1,\cdots,n$*

$$C^{-1}\left\|\sum_{i=1}^{j} e_i\right\| \leqq \left\|\sum_{i=1}^{j} f_i\right\| \leqq C\left\|\sum_{i=1}^{j} e_i\right\|.$$

Thus we have for all $a_i \in R$, $i=1,\cdots,n$

$$(8C\log n)^{-1}\left\|\sum_{i=1}^{n} a_i e_i\right\| \leqq \left\|\sum_{i=1}^{n} a_i f_i\right\| \leqq 8C\log n\left\|\sum_{i=1}^{n} a_i e_i\right\|.$$

We skip the simple proof.

Proof of Theorem 1.1. We consider a map $A \in L(E,F)$ with $\|A\|=1$ and $\|A^{-1}\| \leqq C$. Moreover, we consider $k, l \in N$ (resp. $k', l' \in N$) given by Lemma 1.2 for the matrix

$$(\langle A(e_j), f_i^*\rangle)_{i,j=1}^n \qquad (\text{resp. } (\langle A^{-1}(f_j), e_i^*\rangle)_{i,j=1}^n).$$

We show that the numbers k, l, k', l' are uniformly bounded by a constant that depends only on $d(E,F)$ and r. Thus we get that some diagonal elements of the matrices must be big.

By assumption and by Lemmas 1.7 and 1.8 we get

$$n^r \leqq d(E, l_n^2) \leqq 64C^{*2t}(klk'l')^{2t\beta}(\log n)^2. \tag{1.19}$$

We consider two cases, $t=1$ and $t \geqq 2$. If $t=1$ we observe by Lemma 1.2

$$C^{-1}\tfrac{1}{4}(kl)^{-2}2^{k+l} \leqq n \quad \text{and} \quad \tfrac{1}{4}C^{-1}(k'l')^{-2}2^{k'+l'} \leqq n.$$

On the other hand, we have $n \leqq 2^{k+l+k'+l'}$. Thus we get from (1.19)

$$(\tfrac{1}{16}C^{-2}(klk'l')^{-2}2^{k+l+k'+l'})^{r/2} \leqq 64C^{*2}(klk'l')^{2\beta}(k+l+k'+l')^2.$$

Obviously, k, l, k', l' are bounded by a constant depending only on r and $d(E, l_n^2)$. If $t \geqq 2$ we have $2^{(t-1)(k+l+k'+l')} < n \leqq 2^{t(k+l+k'+l')}$. Thus we get from (1.19)

$$2^{(t-1)r(k+l+k'+l')} \leqq 64C^{*2t}(klk'l')^{2t\beta}t^2(k+l+k'+l')^2.$$

Therefore

$$2^{r(k+l+k'+l')/4} \leqq 8C^*t^{1/t}(klk'l')^{\beta}(k+l+k'+l').$$

Again, it follows that k, l, k', l' are bounded.

Recalling the meaning of k, l, k', l' in Lemma 1.2 we get that for a constant $d = d(r, d(E, l_n^2)) > 0$ there are dn rows

$$(\langle A(e_j), f_i^* \rangle)_{j=1}^n \quad \text{and} \quad (\langle A^{-1}(f_j), e_i^* \rangle)_{j=1}^n$$

each containing at least one coordinate with absolute value greater than d. So, we have mutually disjoint subsets M_t of $\{1, \cdots, n\}$ and indices j_t, $t = 1, \cdots, v$ such that

$$\sum_{t=1}^{v} |M_t| \geqq dn, \tag{1.20}$$

$$|\langle A(e_{j_t}), f_i^* \rangle| \geqq d \qquad \text{for all } i \in M_t. \tag{1.21}$$

Without restriction we may assume that $j_t = t$, $t = 1, \cdots, v$. So we get for $a_1 \geqq a_2 \geqq \cdots \geqq a_{dn} \geqq 0$

$$\|A\| \left\| \sum_{j=1}^{dn} a_j e_j \right\| \geqq \left\| \sum_{i=1}^{n} \left| \sum_{j=1}^{dn} \pm a_j \langle A(e_j), f_i^* \rangle \right| f_i \right\|.$$

By the Khintchine and triangle inequalities we get

$$\sqrt{2} \|A\| \left\| \sum_{j=1}^{dn} a_j e_j \right\| \geqq \left\| \sum_{i=1}^{n} \left(\sum_{j=1}^{v} |a_j \langle A(e_j), f_i^* \rangle|^2 \right)^{1/2} f_i \right\|$$

$$\geqq d \left\| \sum_{j=1}^{v} a_j \sum_{i \in M_j} f_i \right\|.$$

Since we have symmetricity and the first numbers $a_1, \cdots, a_v$ are the greatest we get

$$\sqrt{2} \|A\| d^{-1} \left\| \sum_{j=1}^{nd} a_j e_j \right\| \geqq \left\| \sum_{j=1}^{nd} a_j f_j \right\|.$$

By symmetricity we extend this inequality from nd coordinates to n coordinates. The left hand side inequality is obtained in the same way by considering the matrix $(\langle A^{-1}(f_j), e_i^* \rangle)_{i,j=1}^n$. □

2. On the partial uniqueness of unconditional bases

LEMMA 2.1. *Let $\{e_i\}_{i=1}^n$ be a basis of E such that for some $C > 0$ we have*

$$C^{-1} \left\| \sum_{i=1}^{n} e_i \right\| \leqq \left\| \sum_{i=1}^{n} \varepsilon_i e_i \right\| \leqq C \left\| \sum_{i=1}^{n} e_i \right\| \tag{2.1}$$

for all signs $\varepsilon_i = \pm 1$, $i = 1, \cdots, n$. *Let* $\mathrm{id} \in L(E^*, l_n^2)$ *be the operator* $\mathrm{id}(\sum_{i=1}^n a_i e_i^*) = (a_i)_{i=1}^n$. *Then*

$$\pi_1(\mathrm{id}) \leqq \sqrt{2}\, C \left\| \sum_{i=1}^n e_i \right\|.$$

PROOF. By using the Khintchine inequality we get

$$\begin{aligned} \frac{1}{\sqrt{2}} \sum_{k=1}^n \left(\sum_{i=1}^n |a_i^k|^2 \right)^{1/2} &\leqq \sum_{k=1}^N 2^{-n} \sum_{\varepsilon} \left| \sum_{i=1}^n \varepsilon_i a_i^k \right| \\ &= 2^{-n} \sum_{\varepsilon} \sum_{k=1}^N \left| \left\langle \sum_{i=1}^n a_i^k e_i^*, \sum_{i=1}^n \varepsilon_i e_i \right\rangle \right| \\ &\leqq C \left\| \sum_{i=1}^n e_i \right\| \sup_{\|x\|=1} \sum_{k=1}^n \left| \left\langle \sum_{i=1}^n a_i^k e_i^*, x \right\rangle \right|. \qquad \square \end{aligned}$$

LEMMA 2.2. *Let* $\{e_i\}_{i=1}^n$ *and* $\{f_i\}_{i=1}^n$ *be bases of* E *and* F *with*

$$C_1^{-1} \left\| \sum_{i=1}^n e_i \right\| \leqq \left\| \sum_{i=1}^n \varepsilon_i e_i \right\| \leqq C_1 \left\| \sum_{i=1}^n e_i \right\|, \tag{2.2}$$

$$C_2^{-1} \left\| \sum_{i=1}^n f_i^* \right\| \leqq \left\| \sum_{i=1}^n \varepsilon_i f_i^* \right\| \leqq C_2 \left\| \sum_{i=1}^n f_i^* \right\|, \tag{2.3}$$

for all signs $\varepsilon_i = \pm 1$, $i = 1, \cdots, n$. *Then we have for all* $A \in L(E, F)$

$$\sum_{j=1}^n \left(\sum_{i=1}^n |\langle A(e_i), f_j^* \rangle|^2 \right)^{1/2} \leqq \sqrt{2}\, C_1 C_2 \|A\| \left\| \sum_{i=1}^n e_i \right\| \left\| \sum_{i=1}^n f_i^* \right\|. \tag{2.4}$$

PROOF.

$$\begin{aligned} &\sum_{j=1}^n \left(\sum_{i=1}^n |\langle A(e_i), f_j^* \rangle|^2 \right)^{1/2} \left\| \sum_{i=1}^n f_i^* \right\|^{-1} \\ &\leqq C_2 \left(\sum_{j=1}^n \|(\mathrm{id} \circ A')(f_j^*)\|_2 \right) \left(\sup_{\pm} \left\| \sum_{i=1}^n \pm f_j^* \right\| \right)^{-1} \end{aligned}$$

where $\mathrm{id} \in L(E^*, l_n^2)$ with $\mathrm{id}(\sum_{i=1}^n a_i e_i^*) = (a_i)_{i=1}^n$. Thus we get with Lemma 2.1

$$\begin{aligned} \sum_{j=1}^n \left(\sum_{i=1}^n |\langle A(e_i), f_j^* \rangle|^2 \right)^{1/2} &\leqq C_2 \left\| \sum_{i=1}^n f_i^* \right\| \pi_1(\mathrm{id} \circ A') \\ &\leqq C_2 \left\| \sum_{i=1}^n f_i^* \right\| \|A'\| \pi_1(\mathrm{id}) \\ &\leqq \sqrt{2}\, C_1 C_2 \|A\| \left\| \sum_{i=1}^n f_i^* \right\| \left\| \sum_{i=1}^n e_i \right\|. \qquad \square \end{aligned}$$

LEMMA 2.3. *Let $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ be bases of E and F with*

$$\left\|\sum_{i=1}^n \varepsilon_i e_i\right\|\left\|\sum_{i=1}^n \delta_i e_i^*\right\| \leqq C_1 n, \tag{2.5}$$

$$\left\|\sum_{i=1}^n \varepsilon_i f_i\right\|\left\|\sum_{i=1}^n \delta_i f_i^*\right\| \leqq C_2 n, \tag{2.6}$$

for all signs $\varepsilon_i, \delta_i = \pm 1$, $i = 1, \cdots, n$. Moreover, let $A \in L(E, F)$ be invertible. Then

$$(\sqrt{2}C_1^2C_2\|A^{-1}\|)^{-1}\left\|\sum_{i=1}^n e_i\right\| \leqq \frac{1}{n}\sum_{j=1}^n\left(\sum_{i=1}^n |\langle A(e_i), f_j^*\rangle|^2\right)^{1/2}\left\|\sum_{i=1}^n f_i\right\|$$

$$\leqq \sqrt{2}C_1C_2^2\|A\|\left\|\sum_{i=1}^n e_i\right\|.$$

PROOF. The right hand side inequality is an immediate consequence of Lemma 2.2. In order to prove the left hand side inequality we observe first by Lemma 2.2

$$\sum_{j=1}^n\left(\sum_{i=1}^n |\langle f_j, A^{-1\prime}(e_i^*)\rangle|^2\right)^{1/2} \leqq \sqrt{2}C_1C_2\|A^{-1}\|\left\|\sum_{i=1}^n e_i^*\right\|\left\|\sum_{i=1}^n f_i\right\|. \tag{2.7}$$

Since we have Hölder's inequality

$$1 \leqq \left(\sum_{i=1}^n |\langle A^{-1\prime}(e_i^*), f_j\rangle|^2\right)^{1/2}\left(\sum_{i=1}^n |\langle A(e_i), f_j^*\rangle|^2\right)^{1/2}$$

we get

$$\left(\sum_{j=1}^n\left(\sum_{i=1}^n |\langle A(e_i), f_j^*\rangle|^2\right)^{1/2}\right)\left(\sum_{j=1}^n\left(\sum_{i=1}^n |\langle A^{-1\prime}(e_i^*), f_j\rangle|^2\right)^{1/2}\right) \geqq n^2.$$

By this and (2.7) we get

$$n^2 \leqq \sqrt{2}C_1C_2\|A^{-1}\|\left\|\sum_{i=1}^n e_i^*\right\|\left\|\sum_{i=1}^n f_i\right\|\left(\sum_{j=1}^n\left(\sum_{i=1}^n |\langle A(e_i), f_j^*\rangle|^2\right)^{1/2}\right).$$

This gives the left hand side inequality. □

LEMMA 2.4. *Let $A = \{a_{ij}\}_{i,j=1}^n$ be a $n \times n$-matrix and $A^{-1} = \{b_{ij}\}_{i,j=1}^n$ its inverse. Suppose that for some p, $1 \leqq p < 2$*

$$\left(\sum_{j=1}^n |a_{ij}|^p\right)^{1/p} \leqq C_1 \qquad \text{for all } i = 1, \cdots, n, \tag{2.8}$$

$$\sum_{i=1}^n\left(\sum_{j=1}^n |b_{ji}|^2\right)^{1/2} \leqq C_2 n. \tag{2.9}$$

Then, for all $d>0$ there are $\varepsilon=\varepsilon(d,C_1,C_2)>0$, a subset M of $\{1,\cdots,n\}$ with $|M|\geqq[(1-d)n]$ and a sequence $(i,j(i))$, $i\in M$, such that

$$|a_{i,j(i)}|\geqq\varepsilon \quad \textit{and} \quad |b_{j(i),i}|\geqq\varepsilon.$$

PROOF. We put

$$M=\left\{i\,\middle|\,\left(\sum_{j=1}^{n}|b_{ji}|^2\right)^{1/2}\leqq d^{-1}C_2\right\}.$$

Certainly $|M|\geqq[(1-d)n]$. Together with (2.8) and $\Sigma_{j=1}^{n}|a_{ij}b_{ji}|\geqq 1$ we get the desired result since $p<2$. □

PROPOSITION 2.5. *Let $\{e_i\}_{i=1}^{n}$ and $\{f_i\}_{i=1}^{n}$ be unconditional bases in E and F with $d(E,F)\leqq C$ and*

$$C^{-1}\left\|\sum_{i=1}^{n}e_i\right\|\leqq\left\|\sum_{i=1}^{n}f_i\right\|\leqq C\left\|\sum_{i=1}^{n}e_i\right\|, \tag{2.10}$$

$$\left\|\sum_{i=1}^{n}e_i\right\|\left\|\sum_{i=1}^{n}e_i^*\right\|\leqq Cn, \tag{2.11}$$

$$\left(\sum_{i=1}^{n}|a_i|^p\right)^{1/p}\leqq C\left\|\sum_{i=1}^{n}a_ie_i\right\| \qquad \textit{for some } p,\quad 1\leqq p<2, \tag{2.12}$$

and all $a_i\in R$, $i=1,\cdots,n$. Then, there are constants $C^*=C^*(C,p)>0$, $\delta=\delta(C,p)>0$ and a subset M of $\{1,\cdots,n\}$ with $|M|\geqq\delta n$ and

$$C^{*-1}\left\|\sum_{i\in M}a_ie_i\right\|\leqq\left\|\sum_{i\in M}a_if_i\right\|\leqq C^*\left\|\sum_{i\in M}a_ie_i\right\|$$

for all $a_i\in R$, $i=1,\cdots,n$.

PROOF. We consider $A\in L(F,E)$ with $\|A\|\,\|A^{-1}\|=d(E,F)$. Because of (2.10) and (2.11) we may apply Lemma 2.3. We get

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{n}|\langle A^{-1}(e_i),f_j^*\rangle|^2\right)^{1/2}\leqq C'n.$$

Moreover, by (2.12) we have

$$\|A\|\geqq\|A(f_j)\|\geqq C^{-1}\left(\sum_{i=1}^{n}|\langle A(f_j),e_i^*\rangle|^p\right)^{1/p}.$$

So we can apply Lemma 2.4 for $d=\frac{1}{2}$. We get that there is a subset M of $\{1,\cdots,n\}$ with $|M|\geqq[\frac{1}{2}n]$ and a sequence $(j,i(j))$, $j\in M$, with

$$|\langle A(f_j),e_{i(j)}^*\rangle|\geqq\varepsilon \quad \text{and} \quad |\langle A^{-1}(e_{i(j)}),f_j^*\rangle|\geqq\varepsilon. \tag{2.13}$$

On the other hand we have

$$\alpha_l = \operatorname{card}\{j \mid i(j) = l\} \leqq \varepsilon^{-2} d(E, F).$$

Indeed,

$$d(E,F) = \|A\| \|A^{-1}\| \geqq \|A'(e_l^*)\| \|A^{-1}(e_l)\|$$

$$\geqq \varepsilon^2 \left\| \sum_{i=1}^{\alpha_l} f_i \right\| \left\| \sum_{i=1}^{\alpha_l} f_i^* \right\| \geqq \varepsilon^2 \alpha_l.$$

Therefore, we find even a sequence (i_t, j_t), $t = 1, \cdots, \alpha = [\frac{1}{2}\varepsilon^2 n d(E,F)^{-1}]$ such that

$$|\langle A(f_{j_t}), e_{i_t}^* \rangle| \geqq \varepsilon \quad \text{and} \quad |\langle A^{-1}(e_{i_t}), f_{j_t}^* \rangle| \geqq \varepsilon. \tag{2.14}$$

We have for all signs

$$\|A\| \left\| \sum_{t=1}^{\alpha} a_t f_{j_t} \right\| \geqq \left\| \sum_{t=1}^{\alpha} \pm a_t A(f_{j_t}) \right\|.$$

By the Khintchine inequality and the triangle inequality and (2.14) we get

$$\sqrt{2} \|A\| \left\| \sum_{t=1}^{\alpha} a_t f_{j_t} \right\| \geqq \left\| \sum_{i=1}^{n} \left(\sum_{t=1}^{\alpha} |a_t \langle A(f_{j_t}), e_i^* \rangle|^2 \right)^{1/2} e_i \right\|$$

$$\geqq \left\| \sum_{t=1}^{\alpha} \left(\sum_{t=1}^{\alpha} |a_t \langle A(f_{j_t}), e_{i_t}^* \rangle|^2 \right)^{1/2} e_{i_t} \right\|$$

$$\geqq \varepsilon \left\| \sum_{t=1}^{\alpha} a_t e_{i_t} \right\|.$$

In the same way we get

$$\sqrt{2} \|A^{-1}\| \left\| \sum_{t=1}^{\alpha} a_t e_{i_t} \right\| \geqq \varepsilon \left\| \sum_{t=1}^{\alpha} a_t f_{j_t} \right\|. \qquad \square$$

THEOREM 2.6. *Let $\{e_i\}_{i=1}^n$ be a symmetric basis of E and $\{f_i\}_{i=1}^n$ be an unconditional basis of F with $d(E,F) \leqq C$ and* $\operatorname{cotype}_2(E) \leqq C$. *Suppose that one of the bases $\{e_i\}_{i=1}^n$ or $\{f_i\}_{i=1}^n$ dominates the unit vector basis of l_n^p, $1 \leqq p < 2$. Then there are constants $C^* = C^*(C,p) > 0$, $\varepsilon = \varepsilon(C,p) > 0$ and a subset M of $\{1, \cdots, n\}$ such that $|M| \geqq \varepsilon n$ and*

$$C^{*-1} \left\| \sum_{i \in M} a_i e_i \right\| \leqq \left\| \sum_{i \in M} a_i f_i \right\| \leqq C^* \left\| \sum_{i \in M} a_i e_i \right\|$$

for all $a_i \in R$, $i = 1, \cdots, n$.

PROOF. We have to verify that we are in the situation of Proposition 2.5. Since $\{e_i\}_{i=1}^n$ is a symmetric basis (2.11) is valid. By theorem 3.1 in [2] we have that $\|\sum_{i=1}^n e_i\|$ and $\|\sum_{i=1}^n f_i\|$ are proportional to $\pi_1(E)$ (resp. $\pi_1(F)$). Therefore (2.10) is true and (2.11) is also true for the basis $\{f_i\}_{i=1}^n$. (2.12) is valid by assumption. □

COROLLARY 2.7. *Let $l_n^p \otimes_r l_n^r$ denote the matrix space*

$$\left\| \sum_{i,j=1}^n a_{ij} e_{ij} \right\| = \left(\sum_{j=1}^n \left(\sum_{i=1}^n |a_{ij}|^p \right)^{r/p} \right)^{1/r}, \qquad 1 \leqq p, r < 2.$$

Then $\{\mathrm{sbc}(l_n^p \otimes_r l_n^r) \mid n \in N\}$ is unbounded if and only if $p \neq r$.

PROOF. If $p = r$ we have obviously symmetric bases (with constant 1). Now, suppose $p \neq r$. Observe that for some $C_1 > 0$ we have $\mathrm{cotype}_2(l_n^p \otimes_r l_n^r) \leqq C_1$ for all $n \in N$. It is obvious that for $q = \max\{r, p\}$

$$\left\| \sum_{i,j=1}^n a_{ij} e_{ij} \right\| \geqq \left(\sum_{i,j=1}^n |a_{ij}|^q \right)^{1/q}.$$

Thus we may apply Theorem 2.6. We assume that there is a sequence of spaces F_n, $n \in N$, with symmetric bases $\{f_{ij}\}_{i,j=1}^n$ such that for some $C_2 > 0$ we have

$$d(F_n, l_n^p \otimes_r l_n^r) \leqq C_2.$$

By Theorem 2.6 we get for some $C^* > 0$, $\varepsilon > 0$ and $M \subset \{(i,j) \mid i, j = 1, \cdots, n\}$ with $|M| \geqq \varepsilon n^2$

$$C^{*-1} \left\| \sum_M a_{ij} e_{ij} \right\| \leqq \left\| \sum_M a_{ij} f_{ij} \right\| \leqq C^* \left\| \sum_M a_{ij} e_{ij} \right\|.$$

We find at least one i_0 and j_0 such that

$$\mathrm{card}\{M \cap \{(i_0, j) \mid j = 1, \cdots, n\}\} \geqq \varepsilon n,$$

$$\mathrm{card}\{M \cap \{(i, j_0) \mid i = 1, \cdots, n\}\} \geqq \varepsilon n.$$

By this and the symmetricity of $\{f_{ij}\}_{i,j=1}^n$ we get a contradiction. □

REFERENCES

1. D. J. H. Garling and Y. Gordon, *Relations between some constants associated with finite dimensional Banach spaces*, Israel J. Math. **9** (1971), 346–361.

2. W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri, *Symmetric structures in Banach spaces*, Mem. Amer. Math. Soc., Vol. 19, No. 217, Providence, Rhode Island, May 1979.

3. D. R. Lewis and P. Wojtaszczyk, *Symmetric bases in Minkowski spaces*, Compositio Math. **32** (1976), 293–300.

4. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I, II*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vols. 92 and 97, Springer-Verlag, 1977 and 1979.

DEPARTMENT OF MATHEMATICS
THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL
AND
MATHEMATISCHES INSTITUT
JOHANNES-KEPLER-UNIVERSITÄT
LINZ/DONAU, AUSTRIA